EE120 - Fall'19 - Lecture 11 Notes[1]

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Two Dimensional (2D) Fourier Transform

2D CTFT Analysis Equation:

$$\boxed{X(j\omega_1, j\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2} \tag{1}$$

2D CTFT Synthesis Equation:

$$\boxed{x(t_1,t_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\omega_1, j\omega_2)e^{j\omega_1 t_1}e^{j\omega_2 t_2}d\omega_1 d\omega_2} \tag{2}$$

2D DTFT Analysis Equation:

$$\boxed{X(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x[n_1,n_2]e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}} \tag{3}$$

Note that this is periodic with period $(2\pi, 2\pi)$:

$$X(e^{j\omega_1}, e^{j\omega_2}) = X(e^{j(\omega_1+2\pi)}, e^{j\omega_2}) = X(e^{j\omega_1}, e^{j(\omega_2+2\pi)}).$$

2D DTFT Synthesis Equation:

$$\boxed{x[n_1,n_2] = \frac{1}{(2\pi)^2}\int_{2\pi}\int_{2\pi} X(e^{j\omega_1}, e^{j\omega_2})e^{j\omega_1 n_1}e^{j\omega_2 n_2}d\omega_1 d\omega_2} \tag{4}$$

Absolute integrability/summability conditions for convergence:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x(t_1,t_2)|dt_1 dt_2 < \infty \quad \text{(continuous time)} \tag{5}$$

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} |x[n_1,n_2]| < \infty \quad \text{(discrete time).} \tag{6}$$

Example: $x[n_1,n_2] = \delta[n_1,n_2] := \delta[n_1]\delta[n_2]$.

$$X(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} \delta[n_1,n_2]e^{-j\omega_1 n_1}e^{-j\omega_2 n_2} = e^{-j\omega_1 0}e^{-j\omega_2 0} = 1$$

Example: $x[n_1,n_2] = a^{n_1}b^{n_2}u[n_1,n_2]$, $|a| < 1$, $|b| < 1$, where $u[n_1,n_2] := u[n_1]u[n_2]$.

$$\begin{aligned} X(e^{j\omega_1}, e^{j\omega_2}) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{n_1}b^{n_2}e^{-j\omega_1 n_1}e^{-j\omega_2 n_2} \\ &= \sum_{n_1=0}^{\infty} a^{n_1}e^{-j\omega_1 n_1}\sum_{n_2=0}^{\infty} b^{n_2}e^{-j\omega_2 n_2} \\ &= \frac{1}{1-ae^{-j\omega_1}}\frac{1}{1-be^{-j\omega_2}} \end{aligned}$$

Separability Property of the 2D DTFT:

If $x[n_1, n_2] = x_1[n_1]x_2[n_2]$ then $X(e^{j\omega_1}, e^{j\omega_2}) = X_1(e^{j\omega_1})X_2(e^{j\omega_2})$ as in the examples above. A similar property holds for the 2D CTFT.

Proof:

$$\begin{aligned} X(e^{j\omega_1}, e^{j\omega_2}) &= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x_1[n_1]x_2[n_2]e^{-j\omega_1 n_1}e^{-j\omega_2 n_2} \\ &= \underbrace{\sum_{n_1=-\infty}^{\infty} x_1[n_1]e^{-j\omega_1 n_1}}_{=X_1(e^{j\omega_1})} \underbrace{\sum_{n_2=-\infty}^{\infty} x_2[n_2]e^{-j\omega_2 n_2}}_{=X_2(e^{j\omega_2})} \end{aligned}$$

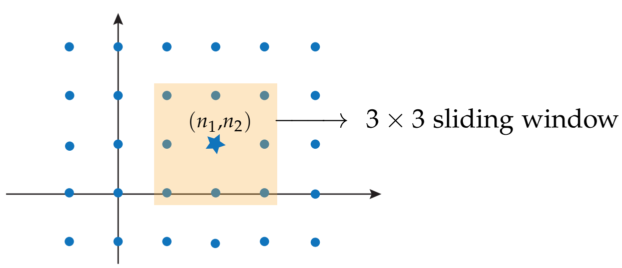
2D Systems

$$x[n_1, n_2] \rightarrow \boxed{} \rightarrow y[n_1, n_2]$$

When the input is $\delta[n_1, n_2]$ the output is called the *impulse response* and denoted $h[n_1, n_2]$ as in 1D systems.

Example: 2D moving average filter

$$y[n_1, n_2] = \frac{1}{9} \sum_{k_1=-1}^{1} \sum_{k_2=-1}^{1} x[n_1 - k_1, n_2 - k_2]$$



$$h[n_1, n_2] = \begin{cases} \frac{1}{9} & -1 \leq n_1 \leq 1 \text{ and } -1 \leq n_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

2D Convolution:

If the system is linear shift-invariant, then:

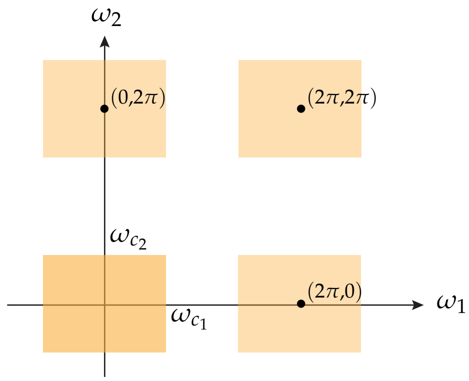
$$\begin{aligned} y[n_1, n_2] &= h[n_1, n_2] * x[n_1, n_2] \\ &= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} h[m_1, m_2]x[n_1 - m_1, n_2 - m_2] \\ &= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} x[m_1, m_2]h[n_1 - m_1, n_2 - m_2]. \end{aligned}$$

Convolution Property of the 2D DTFT

$$\boxed{h[n_1,n_2] * x[n_1,n_2] \longleftrightarrow H(e^{j\omega_1},e^{j\omega_2})X(e^{j\omega_1},e^{j\omega_2})} \tag{7}$$

Example: 2D separable ideal low pass filter

$H(e^{j\omega_1},e^{j\omega_2}) = 1$ in the shaded regions of the (ω_1,ω_2)-plane below and $= 0$ otherwise:



We can write this frequency response as:

$$H(e^{j\omega_1},e^{j\omega_2}) = H_1(e^{j\omega_1})H_2(e^{j\omega_2})$$

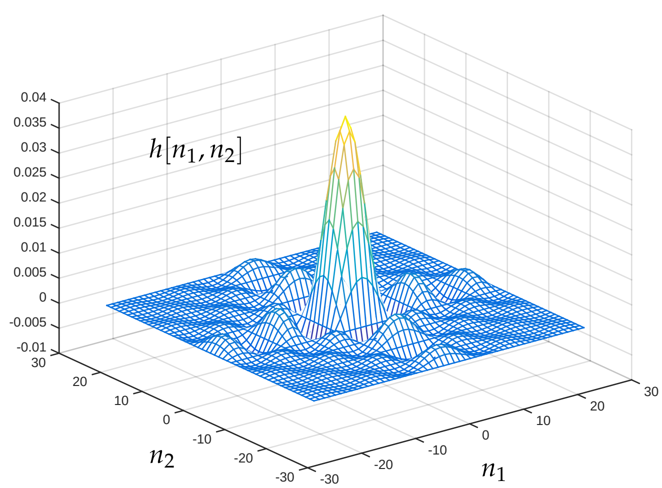
where

$$H_i(e^{j\omega_i}) = \begin{cases} 1 & |\omega_i| \leq \omega_{c_i} \\ 0 & \omega_{c_i} < |\omega_i| \leq \pi \end{cases} \qquad i = 1,2.$$

Then, from the separability property,

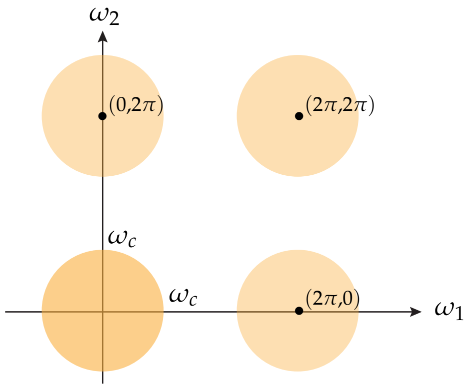
$$h[n_1,n_2] = h_1[n_1]h_2[n_2] = \frac{\omega_{c_1}}{\pi}\text{sinc}\left(\frac{\omega_{c_1}}{\pi}n_1\right)\frac{\omega_{c_2}}{\pi}\text{sinc}\left(\frac{\omega_{c_2}}{\pi}n_2\right)$$

which is depicted below for $\omega_{c_1} = \omega_{c_2} = 0.2\pi$.



Example: 2D circularly symmetric ideal low pass filter

$H(e^{j\omega_1}, e^{j\omega_2}) = 1$ in the shaded regions of the (ω_1, ω_2)-plane below and $= 0$ otherwise:



In the region $[-\pi, \pi] \times [-\pi, \pi]$, this can be expressed as:

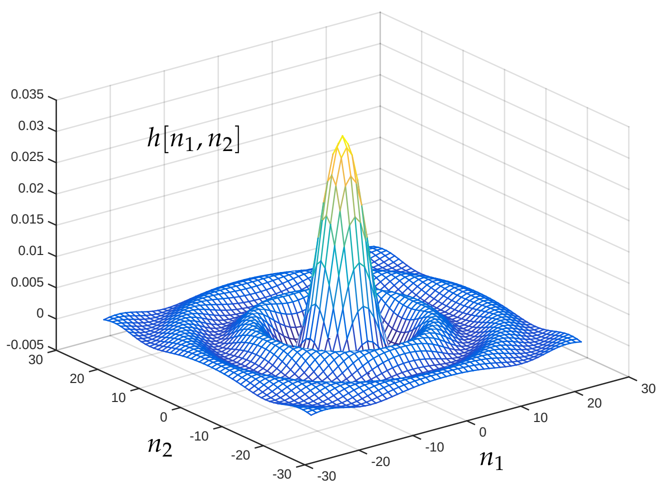
$$H(e^{j\omega_1}, e^{j\omega_2}) = \begin{cases} 1 & \sqrt{\omega_1^2 + \omega_2^2} \leq \omega_c \\ 0 & \sqrt{\omega_1^2 + \omega_2^2} > \omega_c. \end{cases}$$

The 2D DTFT Synthesis Equation yields:

$$h[n_1, n_2] = \frac{\omega_c}{2\pi\sqrt{n_1^2 + n_2^2}} J_1\left(\omega_c \sqrt{n_1^2 + n_2^2}\right)$$

where $J_1(\cdot)$ is the Bessel function of the first kind and first order.[2]

Note that $h[n_1, n_2]$ is not separable. However, like the frequency response $H(e^{j\omega_1}, e^{j\omega_2})$, it exhibits circular symmetry. See the figure below for a depiction of $h[n_1, n_2]$ for $\omega_c = 0.2\pi$.



[2] See mathworld.wolfram.com for a description of Bessel functions of the first kind. The Matlab command to evaluate $J_1(\cdot)$ is `besselj(1,·)` where the first argument specifies the order.

2D DFT

Consider a 2D finite-length signal such that $x[n_1, n_2] = 0$ when $n_1 \notin \{0, 1, \ldots, N_1 - 1\}$ or $n_2 \notin \{0, 1, \ldots, N_2 - 1\}$. The DFT is defined similarly to the 1D case.

Analysis Equation: For $0 \leq k_1 \leq N_1 - 1$ and $0 \leq k_2 \leq N_2 - 1$:

$$\boxed{X[k_1, k_2] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1, n_2] e^{-j\frac{2\pi}{N_1}k_1 n_1} e^{-j\frac{2\pi}{N_2}k_2 n_2}} \tag{8}$$

Synthesis Equation: For $0 \leq n_1 \leq N_1 - 1$ and $0 \leq n_2 \leq N_2 - 1$:

$$\boxed{x[n_1, n_2] = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X[k_1, k_2] e^{j\frac{2\pi}{N_1}k_1 n_1} e^{\frac{2\pi}{N_2}k_2 n_2}} \tag{9}$$

Note, as in the 1D case, that the DFT consists of samples of the DTFT:

$$X[k_1, k_2] = X(e^{j\omega_1}, e^{j\omega_2})|_{\omega_1 = \frac{2\pi}{N_1}k_1, \omega_2 = \frac{2\pi}{N_2}k_2}.$$

Reading for Interested Students

Chapter 27: Data Compression in *The Scientist and Engineer's Guide to Digital Signal Processing* (`www.dspguide.com`). See in particular Figures 27-9, 27-10, 27-11, 27-12, 27-15 illustrating JPEG compression.